Solutions to tutorial exercises for stochastic processes

T1. (a) Let $x \neq u$, then $c(x,\eta) = c(x,\eta_u)$ for all $\eta \in S$. So we have $\gamma(x,u) = \sup_\eta |c(x,\eta) - \eta(x,\eta_u)| = 0$ and also $M = \sup_x \sum_{u \neq x} \gamma(x,u) = 0 < \infty$. Define the generator

$$\mathcal{L}f(\eta) = \sum_{x \in V} c(x,\eta)\big(f(\eta_x) - f(\eta)\big)$$

on the domain

$$\mathcal{D}(\mathcal{L}) = \left\{ f \in C(S) \,:\, \sum_{x \in V} \sup_\eta |f(\eta_x) - f(\eta)| < \infty \right\}.$$

Then it follows from Theorem 5.2 that $\bar{\mathcal{L}}$ is a probability generator. So there exists a spin system with rate function $c(x,\eta)$.

(b) Let $(x_i)_{i \in \mathbb{N}}$ be an enumeration of V and consider $V_n = \{x_1, \ldots, x_n\}$. Similar to (a) there exists a spin system $\eta_t^n \in \{0,1\}^{S_n}$ with rate function $c(x,\eta)$, so that $M = 0$ and

$$\varepsilon_n = \inf_{1 \leq i \leq n,\, \eta} \big(c(x_i,\eta) - c(x_i,\eta_{x_i})\big) = \min_{1 \leq i \leq n} \beta_{x_i} + \delta_{x_i} > 0.$$

It follows that the spin system η_t^n is ergodic. Furthermore the unique stationary measure is given by μ_n under which the state of all x_i, $1 \leq i \leq n$, is Bernoulli distributed with parameter $\frac{\beta_{x_i}}{\beta_{x_i} + \delta_{x_i}}$, independently of each other. This follows from the following calculation:

$$\begin{aligned}
\int \mathcal{L}_n f(\eta) \mu_n(\mathrm{d}\eta) &= \sum_{i=1}^n \int \big(\beta_{x_i} \mathbb{1}_{\{\eta(x_i)=0\}} + \delta_{x_i} \mathbb{1}_{\{\eta(x_i)=1\}}\big)\big(f(\eta_{x_i}) - f(\eta)\big)\mu_n(\mathrm{d}\eta) \\
&= \sum_{i=1}^n \frac{\delta_{x_i}\beta_{x_i}}{\beta_{x_i} + \delta_{x_i}} \left(\sum_{\eta : \eta(x_i)=0} \big(f(\eta_{x_i}) - f(\eta)\big) + \sum_{\eta:\eta(x_i)=1} \big(f(\eta_{x_i}) - f(\eta)\big) \right) \\
&= 0.
\end{aligned}$$

Let μ be the distribution under which the state of all $x \in V$ is Bernoulli distributed with parameter $\frac{\beta_x}{\beta_x + \delta_x}$, independently of each other, so that $\mu|_{S_n} = \mu_n$. Consider the spin system $(\eta_t)_{t \geq 0}$ with initial distribution ν. Let $\eta_t^n = \eta_t|_{S_n}$, then (η_t^n) is an ergodic spin system and

$$\mathbb{P}^\nu_{\eta_t^n} \to \mu_n \quad \text{weakly as } t \to \infty.$$

Since $\bigcup_n S_n = S$ and S is compact it follows from the convergence from the finite-dimensional distributions $\mathbb{P}^\nu_{\eta_t^n}$ that the measures on the entire S also converge, i.e.,

$$\mathbb{P}^\nu_{\eta_t} \to \mu \quad \text{weakly as } t \to \infty.$$

T2. (a) Let D be the maximum degree of the graph. Since $c(x,\eta)$ is determined by the configuration of neighbours of x, we have that $a(x,y) = 0$ whenever $x \neq y$ and $x \nsim y$. It follows that

$$M := \sup_{x\in V} \sum_{y\neq x} a(x,y) = \sup_{x\in V} \sum_{y\neq x, x\sim y} \sup_{\eta\in S} |c(x,\eta_y) - c(x,\eta)| \leq 2D^2 < \infty.$$

From Theorem 5.2 we conclude that there is a spin system with rate function $c(x,\eta)$.

(b) Consider $V = \mathbb{Z}^d$. Consider the 'checkerboard' configuration $\eta \in S$ given by

$$\eta(x) = \|x\|_1 \mod 2.$$

Let $\xi = 1 - \eta$. Then $c(x,\eta) = c(x,\xi) = 0$ for all $x \in V$. It follows that δ_η and δ_ξ are two distinct stationary measures, so that the spin system is not ergodic.

Now let $V = \mathbb{Z}/m\mathbb{Z}$ for m even. We again consider the configurations η given by

$$\eta(x) = x \mod 2,$$

and $\xi = 1 - \eta$. Then, since m is even, $c(x,\eta) = c(x,\xi) = 0$ for all $x \in V$. It follows that δ_η and δ_ξ are two distinct stationary measures, so that the spin system is not ergodic.

Now consider $V = \mathbb{Z}/m\mathbb{Z}$ for m odd. We can now no longer define the checkerboard configuration, since the neighbours $m-1$ and 0 would receive the same spin. Instead, we will show that this spin system is in fact ergodic. For a configuration η, let $n(\eta)$ be the number of edges for which the vertices have different spin:

$$n(\eta) := \sum_{x\in V} \mathbb{1}\{\eta(x) \neq \eta(x+1)\}.$$

Observe that $n(\eta_t) \geq n(\eta_s)$ for all $t \geq s$. We claim that the unique stationary measure is the uniform measure on the set of configurations for which $n(\eta) = m-1$. We define

$$\eta^k := (0,1,0,\dots,1,0,1,1,0,\dots,1), \quad \xi^k := (1,0,1,\dots,0,1,1,0,1,\dots,0),$$

where k is the vertex for which $\eta^k(k) = \eta^k(k+1)$ and $\xi^k(k) = \xi^k(k+1)$. We define

$$\pi := \frac{1}{2m} \sum_{k=0}^{m-1} \delta_{\eta^k} + \delta_{\xi^k}.$$

The measure π is stationary by T10.3, since

$$\begin{aligned}
&\int \mathcal{L}f(\eta)\mathrm{d}\pi = \int \sum_{x\in V} c(x,\eta)(f(\eta_x) - f(\eta))\mathrm{d}\pi \\
&= \frac{1}{2m} \sum_{k=0}^{m-1} \sum_{x\in V} c(x,\eta^k)(f(\eta^k_x) - f(\eta^k)) + c(x,\xi^k)(f(\xi^k_x) - f(\xi^k)) \\
&= \frac{1}{2m} \sum_{k=1}^{m-1} f(\eta^k_k) - f(\eta^k) + f(\eta^k_{k+1}) - f(\eta^k) + f(\xi^k_k) - f(\xi^k) + f(\xi^k_{k+1}) - f(\xi^k) \\
&= 0,
\end{aligned}$$

since $\eta^k_k = \eta^{k-1}$, $\xi^k_k = \xi^{k-1}$, $\eta^k_{k+1} = \eta^{k+1}$ and $\xi^k_{k+1} = \xi^{k+1}$. It remains to show that π is the only stationary measure. Since V is finite, the spin system is a continuous time Markov chain. The set of configurations

$$A := \{\eta \,:\, n(\eta) = m-1\} = \bigcup_{k=0}^{m-1} \{\eta^k, \xi^k\},$$

is an absorbing set for the Markov chain. Furthermore, the Markov chain restricted to A is irreducible. It follows that the restricted chain has a unique stationary measure, which then must be π. Finally, since A is the set of absorbing configurations, any stationary measure cannot put weight outside of A. It follows that π is the unique stationary measure for the complete process.

T3. Let $\mathcal{L}$ be the generator of the coupled spin system with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ f \in C\left(S^2\right) \,:\, \sum_{x\in V} \sup_{(\eta,\zeta)} |f(\eta_x,\zeta) - f(\eta,\zeta)| + |f(\eta,\zeta_x) - f(\eta,\zeta)| + |f(\eta_x,\zeta_x) - f(\eta,\zeta)| < \infty \right\}.$$

For $f \in \mathcal{D}(\mathcal{L})$ we can write

$$\begin{aligned}\mathcal{L}f(\eta,\zeta) = \sum_{x\in V} &\tilde{c}_1(x,\eta,\zeta)\big(f(\eta_x,\zeta) - f(\eta,\zeta)\big) \\ &+ \tilde{c}_2(x,\eta,\zeta)\big(f(\eta,\zeta_x) - f(\eta,\zeta)\big) + \tilde{c}_3(x,\eta,\zeta)\big(f(\eta_x,\zeta_x) - f(\eta,\zeta)\big),\end{aligned}$$

where $\tilde{c}_1, \tilde{c}_2$ and $\tilde{c}_3$ are the rates at which (η,ζ) changes to $(\eta_x,\zeta), (\eta,\zeta_x)$ and (η_x,ζ_x) respectively. These are given by

$$\begin{aligned}\tilde{c}_1(x,\eta,\zeta) &= \mathbb{1}_{\{\eta(x)=0,\zeta(x)=1\}} c_1(x,\eta) + \mathbb{1}_{\{\eta(x)=1\}}\big(c_1(x,\eta) - c_2(x,\zeta)\big),\\ \tilde{c}_2(x,\eta,\zeta) &= \mathbb{1}_{\{\zeta(x)=0\}}\big(c_2(x,\zeta) - c_1(x,\eta)\big) + \mathbb{1}_{\{\eta(x)=0,\zeta(x)=1\}}\big(c_1(x,\eta) - c_2(x,\zeta)\big),\\ \tilde{c}_3(x,\eta,\zeta) &= \mathbb{1}_{\{\zeta(x)=0\}} c_1(x,\eta) + \mathbb{1}_{\{\eta(x)=1\}} c_2(x,\zeta).\end{aligned}$$